Enrollment No: ______________________ **Exam Seat No:**______________________

C.U.SHAH UNIVERSITY

Winter Examination- 2022

Subject Name: Group Theory

Subject Code: 5SC02GRT1 **Branch: M.Sc. (Mathematics)**

Semester: 2 **Date: 20/09/2022** **Time: 11:00 To 02:00** **Marks: 70**

Instructions:

(1) Use of Programmable calculator and any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

SECTION – I

Q-1 **Attempt the Following questions** **(07)**

a. Define: Group **1**
b. Define: Cyclic group. **1**
c. Define: Coset. **1**
d. If $(G,*)$ is a group and $a\,,b \in G$ then prove that $(a*b)^{-1} = b^{-1} * a^{-1}$ **2**
e. Let G be a group. If $(ab)^2 = a^2b^2\ \forall\ a, b \in G$ then prove that G is abelian. **2**

Q-2 **Attempt all questions** **(14)**

a) If $G = \left\{\begin{bmatrix} a & a \\ a & a \end{bmatrix} \ /\ a \text{ is non zero real number}\right\}$ then prove that G is group under multiplication. **5**
b) If $(G,*)$ be a group and $o(a) = n$ then prove the following **5**
(i) $o(a^q) \le o(a)$ for any integer q.
(ii) $o(a^{-1}) = o(a)$.
c) If H is a subgroup of a group$G$ then prove that the set $x^{-1}Hx$ is a subgroup of G for $x \in G$. **4**

OR

Q-2 **Attempt all questions** **(14)**

d) Define: Quaternion group. Prove that it is group under multiplication. **5**
e) Prove that any two right cosets of a subgroup are either disjoint or identical. **5**
f) Prove that the group of prime order is cyclic. **4**

Q-3 **Attempt all questions** **(14)**

a) State and prove Lagrange's theorem **5**
b) If a cyclic subgroup T of G is normal in G then prove that every subgroup of T is normal in G. **5**
c) Define: Congruence relation. Prove that the congruent relation is an equivalent relation. **4**

OR

Q-3 **a)** Define quaternion group. Prove that it is group under multiplication. **5**

b) Let K is a subgroup and H is normal subgroup of a group G. Then prove that $K \cap H$ is normal subgroup of K. **5**

c) How many generators are there of the cyclic group of order 8? **4**

SECTION – II

Q-4 **Attempt the Following questions** **(07)**

a. Define: $p-$ sylow subgroup. **1**

b. Define: Internal Direct product. **1**

c. Define: Conjugate of element in group. **1**

d. Let G be a group with order 12. Does there exists a subgroup of order 5? Justify your answer. **2**

e. Define: Cycle decomposition. **2**

Q-5 **Attempt all questions** **(14)**

a) State and prove fundamental theorem of homomorphism. **7**

b) List all conjugate classes in S_3, find $c_a^{'}$s for each class. **5**

c) Let G be a group and ϕ is an automorphism of G. If $a \in G$ is of order $o(a) = n > 0$ then prove that $o(\phi(a)) = o(a)$. **2**

OR

Q-5 **a)** Prove that each permutation $f \in S_n$ can be expressed as a composition of disjoint cycles. **6**

b) Let G be group. For a fixed element g in G, define $\phi : G \to G$ by $\phi(x) = gxg^{-1}$. Prove that is ϕ an isomorphism of G on to G. **5**

c) Prove that the order of the set of all even permutation in S_n is $\frac{n!}{2}$ for $n \geq 2$. **3**

Q-6 **Attempt all questions** **(14)**

a) State and prove Cayley's theorem. **7**

b) If G_1 and G_2 are groups then prove that sets $G_1 \times \{e_2\}$, $\{e_1\} \times G_2$ are normal subgroups of $G_1 \times G_2$. Also prove that $G_1 \times \{e_2\} \cong G_1$ and $\{e_1\} \times G_2 \cong G_2$. **5**

c) If order of group is 49, then prove that G is abelian. **2**

OR

Q-6 **Attempt all Questions**

a) State and prove Sylow's theorem **7**

b) Let G be a group and suppose G is the internal direct product of $N_1, N_2, \dots, N_n$. Let $T = N_1 \times N_2 \times \dots \times N_n$. Then prove that G and T are isomorphic. **5**

c) How many conjugate classes in S_4? **2**